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# Quark mixing in the standard model and space rotations

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**Abstract.** The rotation matrix and the Cabibbo–Kobayashi–Maskawa (CKM) matrix are discussed. The CKM matrix is viewed as the rotation matrix in Euler angles with pitch–roll–yaw convention for the angles and as the angle–axis representation of the rotation matrix. A comparison of the exponential parameterisation of the CKM matrix with the matrix exponent generator of the space rotations is made. How to account for the $CP$ violating phase in CKM and the O(3) rotation matrix in the angle–axis form is discussed in the context of such a view of the mixing matrix. The generation of the new parameterisations of the CKM matrix in an exponential form with distinguished $CP$ violating part is demonstrated.

## 1 Introduction

The standard model of particle physics [1–3] is a theoretical framework that accounts for the observed data and describes the electromagnetic and weak interactions of particles by a gauge theory, based on the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ group and spontaneous symmetry breaking via the Higgs mechanism [1, 2, 4]. The extensive experimental activity carried out during the last decades and aimed on testing of the SM proved its reliability and coherence. It has been extremely successful in predictions of a wide range of phenomena and at present the only missing particle to be confirmed experimentally remains the Higgs boson. The standard model has been widely discussed in the literature, so we will not dedicate much space here to the description of the SM itself, and we will focus on the important role that the quark mass mixing matrix plays in it.

The quark mass eigenstates $(q)$ are different from the weak interaction quark eigenstates $(q')$, which are a linear superposition of the mass eigenstates. In the framework of the SM the weak charged interactions of hadrons can be described by the following Lagrangian:

$$L_{\mathrm{int}\, J^c W} = \frac{g}{\sqrt{2}}\left(W_\alpha^+ J^{C\alpha} + W_\alpha J^{C\alpha\,+}\right), \tag{1}$$

where $g = e/\sin\theta_{\mathrm{W}}$ is the coupling constant, $\theta_{\mathrm{W}}$ is the Weinberg angle, $W_\alpha$ are the $W^\pm$ charged boson fields and $J^{C\alpha}$ is the hadronic charged current

$$J^{C\alpha} = \bar{U}_i V_{ik} \gamma^\alpha (1+\gamma_5) D_k\,, \tag{2}$$

which links the vector of the $2e/3$ charged quarks $U = (u, c, t)$ with the $-1e/3$ charged quark vector $D = (d, s, b)$ with the coupling constants $V_{ik}$. Cabibbo postulated on the basis of the experimental data, regarding just two generations of quarks, that the weak eigenstates were not the flavour eigenstates of the strong interaction, but a linear combination, rotated by an angle $\theta$, which took the name of Cabibbo angle. Thus, for the two quark doublets, completed with the charm quark, the hadronic charged current can be written as follows:

$$J^{+\alpha} = \begin{pmatrix}\bar{u} & \bar{c}\end{pmatrix} \begin{pmatrix}\cos\theta_{\mathrm{C}} & \sin\theta_{\mathrm{C}} \\ -\sin\theta_{\mathrm{C}} & \cos\theta_{\mathrm{C}}\end{pmatrix} \gamma^\alpha (1+\gamma_5) \begin{pmatrix} d \\ s \end{pmatrix}. \tag{3}$$

Equation (3) contains a real unitary matrix, which gives a rotation in two dimensions in the angle $\theta_{\mathrm{C}}$. This matrix is sometimes called the Cabibbo matrix. With the discovery of the third generation of quarks, this matrix was generalised by Kobayashi and Maskawa [5] to a $3\times 3$ unitary mixing matrix. Usually, the quark mixing is expressed via this matrix $\mathbf{V}$, called Cabibbo–Kobayashi–Maskawa (CKM) matrix, which by agreement acts on the charge $-e/3$ physical mass states $(d, s, b)$ and transforms them into new interaction eigenstates $(d', s', b')$; thus we have

$$|q'\rangle = \mathbf{V}\,|q\rangle \Rightarrow \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{4}$$

In practice, the individual values of the entries of the mixing matrix can be determined on the basis of the experimental data of the weak decays of the relevant quarks (see, for example [6]) and from the analysis of the deep inelastic scattering of the neutrinos (see, for example, the citations

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in [7]). The $CP$ violation processes involve the complex phase in the CKM matrix elements and it is assumed that the observed $CP$ violation is related solely to the non-zero value of this phase. The principal errors for the determination of the values for most of the CKM matrix elements come from both experimental and theoretical uncertainties related to the interpretation of the data, whereas the constraints of unitarity relate different elements in the quark mass mixing matrix to each other.

## 2 Standard and exponential parameterisations of the CKM matrix and the rotation matrix

Nowadays the standard common form of the CKM matrix comes with a placement of the phase different from the parameterisation originally proposed in [5], and it is given by the three mixing angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and the $CP$ non-invariant phase $\delta$ [7–11]:

$$\mathbf{V} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix}, \tag{5}$$

with $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and the "generation" index $i, j = 1, 2, 3$. With one of the angles vanishing, the mixing between the related generations vanish too. The limit $\theta_{23} = \theta_{13} = 0$ represents the well known case of Cabibbo mixing [12] with $\theta_{12} = \theta_{\mathrm{C}}$, in which the third generation decouples from the other ones.

Despite that no physics can depend on which of the parameterisations is used as long as the single one is used consistently, a thoughtful choice of a proper parameterisation can help in the analysis of the specific problem and may reveal or mask certain features. It is well known that the CKM matrix can be expressed as the product of three real rotation matrices $\mathbf{B}, \mathbf{C}, \mathbf{D}$ and three diagonal matrices $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$:

$$\mathbf{V} = \mathbf{P}_2\mathbf{R_{23}}\mathbf{R_{13}}\mathbf{P}_1\mathbf{R_{12}}\mathbf{P_3}, \tag{6}$$

where the three rotation matrices $\mathbf{R_{12}}$, $\mathbf{R_{23}}$ and $\mathbf{R_{13}}$ are given by three independent rotation angles and

$$\mathbf{P}_2 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\delta} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathbf{P_3} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

The CKM matrix with $\delta = 0$ is in fact the rotation matrix. According to Euler's rotation theorem the rotation is given by the three rotation angles $(\phi, \theta, \psi)$. The standard form of the CKM matrix (5) without the $CP$ term $\delta$ represents the rotation in the "$xyz$" pitch-roll-yaw convention, where the roll $\phi = \theta_{12}$ is the rotation about the $x$-axis, the pitch $\theta = -\theta_{13}$ the rotation about the $y$-axis and the yaw $\psi = \theta_{23}$ the rotation about the $z$-axis. When $\theta = \psi = 0$, the Cabibbo case verifies.

The exponential parameterisation of the CKM matrix also finds its analogy in the classical mechanics of rotations, in particular, the angle–axis presentation and the Euler parameters, in quaternion forms. Indeed, let us consider the exponential representation of the mass mixing matrix [13]:

$$\hat{\mathbf{V}} = \mathrm{e}^{\mathbf{A}}, \tag{8}$$

where $\mathbf{A}$ is a $3\times3$ matrix with additional parameters $\alpha$ and $\beta$, which can be written in the following anti-Hermitian form to ensure the unitarity of the matrix $\mathbf{V}$:

$$\mathbf{A} = \begin{pmatrix} 0 & \lambda & \alpha\lambda^3\mathrm{e}^{\mathrm{i}\delta} \\ -\lambda & 0 & -\beta\lambda^2 \\ -\alpha\lambda^3\mathrm{e}^{-\mathrm{i}\delta} & \beta\lambda^2 & 0 \end{pmatrix}, \tag{9}$$

where the parameter $\delta$ accounts for the violation of the $CP$ symmetry and the parameter $\lambda$ is responsible for the quark flavour mixing, whereas unitarity remains preserved [14].

If we ignore the $CP$ violating term, assuming real entries for the $\mathbf{A}$ matrix, then such CKM matrix reminds one of the angle–axis form of a three dimensional rotation $\mathbf{M}$, defined by a single angle of rotation $\Phi$ and a direction unit vector $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ of the fixed axis, around which the rotation is performed:

$$\mathbf{M}(\hat{\mathbf{n}}, \Phi) = \mathrm{e}^{\Phi\mathbf{N}}, \quad \mathbf{N} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix},$$
$$\hat{\mathbf{n}} = (n_x, n_y, n_z). \tag{10}$$

The explicit expression for the rotation matrix (10) in terms of the single rotation angle and the direction unit vector in 3D space can be written as follows:

$$\mathbf{M}(\hat{\mathbf{n}}, \Phi) = \begin{pmatrix} \cos\Phi + (1-\cos\Phi)\,n_x^2 & (1-\cos\Phi)\,n_x n_y - \sin\Phi n_z & (1-\cos\Phi)\,n_x n_z + \sin\Phi n_y \\ (1-\cos\Phi)\,n_y n_x + \sin\Phi n_z & \cos\Phi + (1-\cos\Phi)\,n_y^2 & (1-\cos\Phi)\,n_y n_z - \sin\Phi n_x \\ (1-\cos\Phi)\,n_z n_x - \sin\Phi n_y & (1-\cos\Phi)\,n_z n_y + \sin\Phi n_x & \cos\Phi + (1-\cos\Phi)\,n_z^2 \end{pmatrix}. \tag{11}$$

The rotation angle $\Phi$ and the unit vector components are related to the elements of the CKM matrix in the standard parameterisation (5) $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ through the following relations:

$$\cos\Phi = \frac{1}{2}(c_{12}c_{13} + c_{12}c_{23} + c_{23}c_{13} - s_{12}s_{23}s_{13} - 1),$$
$$n_x \sin\Phi = \frac{1}{2}(-s_{23}(c_{12} + c_{13}) - s_{12}c_{23}s_{13}), \tag{12}$$
$$n_y \sin\Phi = \frac{1}{2}(s_{13}(1 - c_{12}c_{23}) - s_{12}s_{23}),$$
$$n_z \sin\Phi = \frac{1}{2}(-s_{12}(c_{23} + c_{13}) - c_{12}s_{23}s_{13}). \tag{13}$$

The Cabibbo case (see the comments after (5)) occurs when the unit vector is turned against the $z$-axis, $\hat{\mathbf{n}}_{\text{Cabbibo}} = (0, 0, -1)$, which corresponds to the following exponential mixing matrix:

$$\mathbf{V}_{\text{Cabibbo}} = \exp \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{14}$$

Comparing (10) for the rotation matrix in axis–angle presentation with the exponential quark mass mixing matrix (8) and (9), we find the expression for the single rotation angle $\Phi$ in terms of the parameters $\lambda$, $\alpha$ and $\beta$ of the mixing matrix:

$$\Phi = \lambda\sqrt{\zeta}\,, \quad \zeta = 1+\rho+\tau, \quad \rho \equiv (\beta\lambda)^2\,, \quad \tau \equiv (\alpha\lambda^2)^2\,. \tag{15}$$

The coordinates of the unit vector $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ are determined by the parameter $\lambda = \sin\theta_C$ and the parameters $\alpha$ and $\beta$ of the exponential parameterisation, as follows:

$$\hat{\mathbf{n}} = (n_x, n_y, n_z)\,, \quad n_x = \sqrt{\frac{\rho}{\zeta}}\,, \quad n_y = \sqrt{\frac{\tau}{\zeta}}\,, \quad n_z = -\frac{1}{\sqrt{\zeta}}. \tag{16}$$

If we set the only rotation angle in the axis–angle presentation $\Phi = 0$, then we obtain from any of (10) or (11) the unit matrix for the rotation $\mathbf{M}(\hat{\mathbf{n}}, \Phi)|_{\Phi=0} = \mathbf{I} = \text{diag}\,\{1,1,1\}$.

Thus we may imagine the rotation in the only plane with the normal vector (16), which determines the quark mixing by the single rotation angle, as the Cabibbo angle does for four quarks, and it is enough to set this fundamental angle zero to cancel the rotation from the geometric point of view and, respectively, cancel the mixing between quarks from the physical point of view.

## 3 $CP$ violation in the mixing matrix and the generation of new parameterisations

Let us decompose the exponent (9) of the matrix $\hat{\mathbf{V}}$ in the sum of the real and imaginary parts, $\mathbf{A}_1$ and $\mathbf{A}_2$, as follows[1]:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2\,. \tag{17}$$

$$\mathbf{A}_2 = |\mathbf{A}| = \begin{pmatrix} 0 & \lambda & \alpha\lambda^3 \\ -\lambda & 0 & -\beta\lambda^2 \\ -\alpha\lambda^3 & \beta\lambda^2 & 0 \end{pmatrix}, \tag{18}$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & \alpha\lambda^3(-1+\mathrm{e}^{\mathrm{i}\delta}) \\ 0 & 0 & 0 \\ \alpha\lambda^3(1-\mathrm{e}^{-\mathrm{i}\delta}) & 0 & 0 \end{pmatrix}, \tag{19}$$

[1] The other possibility to divide the real and the imaginary parts would be the following:

$$\mathbf{A}_2 = \lambda \begin{pmatrix} 0 & 1 & \alpha\lambda^2\cos(\delta) \\ -1 & 0 & -\beta\lambda \\ -\alpha\lambda^2\cos(\delta) & \beta\lambda & 0 \end{pmatrix},$$

$$\mathbf{A}_1 = \mathrm{i}\alpha\lambda^3\sin\delta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $\mathbf{A}_2$ is the matrix (9) with $\delta = 0$. Following the analogy between (18) and the argument of the exponent in the matrix of the O(3) rotation in the angle–axis presentation, we note that the transformation $\hat{\mathbf{V}}$ takes the following form:

$$\hat{\mathbf{V}} \cong \mathbf{P}_{\text{Rot}}\mathbf{P}_{CP}\left(1 - \frac{1}{2}[\mathbf{A}_2, \mathbf{A}_1]\right), \tag{20}$$

where the commutator is of the order $O(\lambda^4)$. Thus in this approximation the transformation, corresponding to quark mixing with $CP$ violation, is composed of the purely rotational part

$$\mathbf{P}_{\text{Rot}} = \mathrm{e}^{\mathbf{A}_2} = \exp \begin{pmatrix} 0 & \lambda & \alpha\lambda^3 \\ -\lambda & 0 & -\beta\lambda^2 \\ -\alpha\lambda^3 & \beta\lambda^2 & 0 \end{pmatrix}, \tag{21}$$

which is related to the rotation matrix (10) via (15) and (16) as discussed in the previous sections. The $CP$ violating part $\mathbf{P}_{CP}$ can be written as follows:

$$\mathbf{P}_{CP} = \mathrm{e}^{\mathbf{A}_1} = \begin{pmatrix} \cos 2\Delta & 0 & \kappa^+\sin 2\Delta \\ 0 & 1 & 0 \\ \kappa^-\sin 2\Delta & 0 & \cos 2\Delta \end{pmatrix}, \tag{22}$$

where

$$\Delta = \alpha\lambda^3\sin\frac{\delta}{2}\,, \quad \kappa^{\pm} = \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\frac{\delta}{2}}\,. \tag{23}$$

The $CP$ violating matrix written above reminds one of the Cabibbo mixing matrix (3), acting on the quarks $d$ and $b$ with the weights $\kappa$ for the (1,3) and (3,1) entries:

$$d' = d\cos\Delta + \kappa^+ b\sin\Delta\,, \tag{24}$$

$$b' = \kappa^- d\sin\Delta + b\cos\Delta\,. \tag{25}$$

The transformation (22) preserves the norm

$$\langle d'|d'\rangle = \langle b'|b'\rangle = 1 \tag{26}$$

and orthogonality:

$$\langle d'|b'\rangle = \langle b'|d'\rangle = 0\,. \tag{27}$$

Note that the $CP$ violating matrix can be expressed in terms of the Bessel functions, if we take into account the following well known series expansion:

$$\exp[\mathrm{i}x\sin\alpha] = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\alpha} J_n(x)\,, \tag{28}$$

where $J_n(x)$ are the Bessel functions of the first kind, and we employ the following relevant formulae:

$$\cos(x\sin\alpha) = \sum_{n=-\infty}^{\infty} J_n(x)\cos n\alpha\,, \tag{29}$$

$$\sin(x\sin\alpha) = \sum_{n=-\infty}^{\infty} J_n(x)\sin n\alpha\,. \tag{30}$$

Thus, we can separate the $CP$ violating phase $\delta$ from the other parameters in (22) as follows:

$$\mathbf{P}_{CP} = \begin{pmatrix} \sum_{n=-\infty}^{\infty} J_n(2\Lambda)\cos\left(\frac{n\delta}{2}\right) & 0 & \kappa^+ \sum_{n=-\infty}^{\infty} J_n(2\Lambda)\sin\left(\frac{n\delta}{2}\right) \\ 0 & 1 & 0 \\ \kappa^- \sum_{n=-\infty}^{\infty} J_n(2\Lambda)\sin\left(\frac{n\delta}{2}\right) & 0 & \sum_{n=-\infty}^{\infty} J_n(2\Lambda)\cos\left(\frac{n\delta}{2}\right) \end{pmatrix}, \quad (31)$$

with

$$\Lambda = \alpha\lambda^3 . \quad (32)$$

The Hermitian-conjugated matrices inverse with respect to $P_{CP}$ exist, so that the following identity is verified:

$$\mathbf{P}_{CP}^{-1} \cdot \mathbf{P}_{CP} = \mathbf{P}_{CP}^{+} \cdot \mathbf{P}_{CP} = \mathbf{I} . \quad (33)$$

Moreover, following [14] we rewrite $\hat{\mathbf{V}}$ with the help of the well known formula from the theory of matrices in the following form:

$$\begin{aligned} \hat{\mathbf{V}} &= \mathrm{e}^{\hat{\mathbf{A}}} = \exp\left(\hat{\mathbf{A}}_1 + \hat{\mathbf{A}}_2\right) \\ &= \exp\left(\frac{\hat{\mathbf{A}}_1}{2}\right)\exp\left(\hat{\mathbf{A}}_2\right)\exp\left(\frac{\hat{\mathbf{A}}_1}{2}\right) + \frac{1}{4!}\left[\hat{\mathbf{A}}_1\left[\hat{\mathbf{A}}_1\hat{\mathbf{A}}_2\right]\right] \\ &\quad + o(\lambda^9) . \end{aligned} \quad (34)$$

Equation (34) for the exponential matrix yields in fact the new parameterisation with the mass mixing matrix, which involves the rotation matrix $\mathbf{P}_{\mathrm{Rot}}$, defined by (21), this being the following matrix product:

$$\tilde{\mathbf{V}} = \mathbf{P}_{CP/2} \cdot \mathbf{P}_{\mathrm{Rot}} \cdot \mathbf{P}_{CP/2} , \quad \mathbf{P}_{CP/2} = \exp\left(\frac{\hat{\mathbf{A}}_1}{2}\right) . \quad (35)$$

This matrix $\tilde{\mathbf{V}}$ has the same entries as the matrix $\mathbf{V}$ in (8) at least up to $\lambda^9$ power. The exponent of $\mathbf{P}_{\mathrm{Rot}}$ has real entries, which are just the absolute values of the proper entries of the exponential mixing matrix (9). The matrix $\mathbf{P}_{CP/2}$, accounting for the complex term contribution, i.e. for the $CP$ violating part, can be written as follows:

$$\mathbf{P}_{CP/2} = \begin{pmatrix} \cos\Delta & 0 & \kappa^+ \sin\Delta \\ 0 & 1 & 0 \\ \kappa^- \sin\Delta & 0 & \cos\Delta \end{pmatrix} , \quad (36)$$

where $\Delta$ and $\kappa$ are defined by (23). Equation (36) can also be written in terms of the Bessel functions to distinguish the $CP$ phase $\delta$ as follows:

$$\mathbf{P}_{CP/2} = \begin{pmatrix} \sum_{n=-\infty}^{\infty} J_n(\Lambda)\cos\left(\frac{n\delta}{2}\right) & 0 & \kappa^+ \sum_{n=-\infty}^{\infty} J_n(\Lambda)\sin\left(\frac{n\delta}{2}\right) \\ 0 & 1 & 0 \\ \kappa^- \sum_{n=-\infty}^{\infty} J_n(\Lambda)\sin\left(\frac{n\delta}{2}\right) & 0 & \sum_{n=-\infty}^{\infty} J_n(\Lambda)\cos\left(\frac{n\delta}{2}\right) \end{pmatrix} . \quad (37)$$

A direct check of the unitarity of the matrix $\tilde{\mathbf{V}}$, given by (35) and (36), confirms that the new parameterization $\tilde{\mathbf{V}} = \mathbf{P}_{CP/2} \cdot \mathbf{P} \cdot \mathbf{P}_{CP/2}$ is exactly unitary.

## 4 Conclusions

In the present paper we have discussed the parallel between the CKM matrix in the standard model and geometric rotations in classical mechanics. The geometric nature of the CKM matrix without $CP$ violations effects reduces the problem of quark mixing to the Rodriguez rotation, well known in mechanics. Indeed, in the case of conserved $CP$, the mixing of the quarks in the SM can be viewed as a rotation around a fixed axis in 3D space by the angle $\Phi$. When two of the three mixing angles in the CKM matrix are zero, the rotation angle $\Phi$ coincides with the last non-vanishing angle in the CKM matrix, and we have the Cabibbo case. Setting the rotation angle – the mixing angle for the quarks – $\Phi = 0$, the mixing between the quarks fades out since the mixing matrix becomes the unit matrix $\mathbf{I}$. The presence of a $CP$ violating phase breaks this symmetry and hence such a simple geometric picture gets lost.

Moreover, the $CP$ violation has a certain geometric interpretation in the exponential mass mixing parameterisation. In particular, the "$\mathbf{y}$" component of the rotation direction vector $\mathbf{v}$ in the angle–axis presentation becomes complex. Equations (35) and (36) define a new unitary mixing matrix parameterisation with distinguished $CP$ violating part. The $CP$ violation is effectively separated in the coefficients of the expansion into a series of Bessel functions in (31) and (37).

The $CP$ non-invariant phase in (9) breaks the dynamical symmetry, and this physical situation can no more be described with the help of the ordered product, involving O(3) generators alone. The matrix (19) in addition to (18) complicates the physical picture and a further exact treatment of the problem can be performed on the basis of the theories with extended symmetries, employing the Gell-Mann SU(3) matrices. Although in this case rigorous results can be obtained, the analysis becomes formally cumbersome and the emerging picture is scarcely transparent. On the other hand, the mixing matrix $\mathbf{V}$ in the form found, (35), includes the corrections up to $O(\lambda^9)$ and abundantly satisfies the present level of experimental confidence, so that it is appropriate for the analysis of the experimental data at present and in the near future.

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